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Rheometrical flow systems

Part 1. Flow between concentric spheres rotating about different axes

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The flow of an elastico-viscous liquid contained between two concentric spheres which are rotating with the same angular velocity about axes passing through the centre of the spheres is considered. The angle between these axes is small. The solution is obtained by expanding the velocity components in terms of a small parameter α^2, which is usually associated with problems involving oscillatory flows. The analysis is shown to have a direct application to the Balance Rheometer. In particular, it is shown that inertial effects in this rheometer are likely to be very small.

1. Introduction

When elastico-viscous liquids are subjected to a small sinusoidal deformation, their behaviour can be characterized by equations of state of the form,†

$$p_{ik} = -pg_{ik} + p'_{ik}, \tag{1}$$

$$p'_{ik} = 2\eta^* e^{(1)}_{ik}, \quad (2)‡$$

where p_{ik} is the stress tensor, p an arbitrary isotropic pressure (in the case of an incompressible fluid), g_{ik} the metric tensor of a suitable co-ordinate system and $e^{(1)}_{ik}$ is the (first) rate-of-strain tensor. η^* is known as the complex viscosity and is usually expressed in the form,

$$\eta^* = \eta' - i\frac{G'}{\Omega}, \tag{3}$$

where Ω is the frequency of the oscillation and η' is given the name 'dynamic viscosity' and G' the name 'dynamic rigidity'. Rheologists usually determine η^* by subjecting the liquids to an unsteady motion in which the (Eulerian) velocity components are small and have a factor $e^{i\Omega t}$. Various rheometers have been constructed on this principle and consistent theories for these instruments are available (Walters 1968).

It is, however, not essential to consider an unsteady flow to determine η^*. It is sufficient to generate a flow for which individual fluid elements are subjected

† Covariant suffices are written below, contravariant suffices above, and the usual summation convention for repeated suffices is assumed.

‡ We are assuming here that the rate-of-strain tensor is complex and has some sinusoidal dependence such as $e^{i\Omega t}$, the real part being implied.

to a small sinusoidal deformation. Such a flow may in fact be steady in the sense that $\partial/\partial t \equiv 0$. In this and subsequent papers, we shall consider flow situations of this type.

In the present paper, we consider the situation illustrated in figure 1. The fluid is contained between concentric spheres, the inner sphere having a radius r_1 and the outer sphere a radius r_2. The inner sphere rotates with constant angular velocity Ω about the axis Oz and the outer sphere rotates with the same angular velocity Ω about the axis $O\bar{z}$. The axis $O\bar{z}$ is in the plane Oxz. The angle ϵ between the axes Oz and $O\bar{z}$ is assumed to be small enough for second-order terms in ϵ to

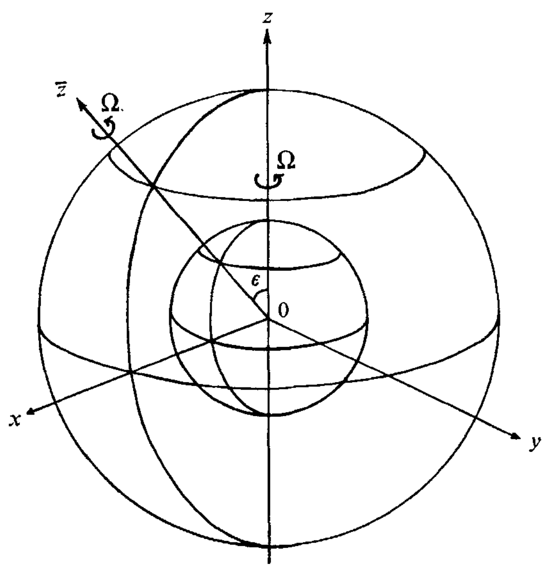

FIGURE 1

be ignored. The purpose of the analysis is to determine the couple exerted by the fluid on the inner sphere. In particular, we shall be concerned with the components of this couple about the axis Ox (C_x) and about the axis Oy (C_y).

A new rheometer has recently been introduced by Kepes (1970). This 'balance rheometer' (soon to be manufactured commercially by Contraves, A.G., Switzerland), subjects the fluid to the type of flow considered in the present paper. In this rheometer, the angle between the axes of rotation is always less than 6° and can be as small as 0·5°; the radii of inner and outer spheres are 2·1 cm and 2·2 cm, respectively, and the maximum speed of rotation is 20 Hz. Kepes claims that the measurement of the couples C_x and C_y can be used to determine η' and G', respectively. In the present paper, we substantiate this claim, and at the same time indicate to what extent inertial effects are likely to modify the interpretation of the experimental results.

2. Basic equations

It is convenient to introduce spherical polar co-ordinates (r, θ, ψ) defined by

$$\left.\begin{aligned} x &= r\sin\theta\cos\psi, \\ y &= r\sin\theta\sin\psi, \\ z &= r\cos\theta. \end{aligned}\right\} \tag{4}$$

It is also useful to introduce a further set of co-ordinates $(r, \bar{\theta}, \bar{\psi})$ related to the $O\bar{z}$ and Oy axes. The following relations hold:

$$\left.\begin{aligned} \cos\bar{\theta} &= \cos\theta\cos\epsilon + \sin\theta\cos\psi\sin\epsilon, \\ \sin\bar{\theta}\cos\bar{\psi} &= \sin\theta\cos\psi\cos\epsilon - \cos\theta\sin\epsilon. \end{aligned}\right\} \tag{5}$$

Using (5), it can be shown that the boundary conditions for the problem under consideration can be expressed in the form,†

$$\left.\begin{aligned} v_{(r)} = 0, \quad v_{(\theta)} = 0, \quad v_{(\psi)} = \Omega r_1 \sin\theta \quad \text{on} \quad r = r_1, \\ v_{(r)} = 0, \quad v_{(\theta)} = -\epsilon\Omega r_2 \sin\psi, \quad v_{(\psi)} = \Omega r_2[\sin\theta - \epsilon\cos\theta\cos\psi] \quad \text{on} \quad r = r_2, \end{aligned}\right\} \tag{6}$$

where $v_{(r)}$, $v_{(\theta)}$, $v_{(\psi)}$ denote the physical components of the velocity vector. In (6) second-order terms in ϵ have been ignored. If the body forces are incorporated in the isotropic pressure p, the stress equations of motion for a steady flow become

$$\rho\left[v_{(r)}\frac{\partial v_{(r)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(r)}}{\partial \theta} + \frac{v_{(\psi)}}{r\sin\theta}\frac{\partial v_{(r)}}{\partial \psi} - \frac{v_{(\theta)}^2}{r} - \frac{v_{(\psi)}^2}{r}\right]$$
$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, p_{(rr)}) + \frac{\partial}{\partial\theta}(r\sin\theta\, p_{(r\theta)}) + \frac{\partial}{\partial\psi}(r p_{(r\psi)})\right] - \frac{p_{(\theta\theta)}}{r} - \frac{p_{(\psi\psi)}}{r}, \tag{7}$$

$$\rho\left[v_{(r)}\frac{\partial v_{(\theta)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(\theta)}}{\partial \theta} + \frac{v_{(\psi)}}{r\sin\theta}\frac{\partial v_{(\theta)}}{\partial \psi} + \frac{v_{(r)}v_{(\theta)}}{r} - \frac{v_{(\psi)}^2\cot\theta}{r}\right]$$
$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, p_{(\theta r)}) + \frac{\partial}{\partial\theta}(r\sin\theta\, p_{(\theta\theta)}) + \frac{\partial}{\partial\psi}(r p_{(\theta\psi)})\right] - \frac{p_{(\psi\psi)}\cot\theta}{r} + \frac{p_{(\theta r)}}{r}, \tag{8}$$

$$\rho\left[v_{(r)}\frac{\partial v_{(\psi)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(\psi)}}{\partial \theta} + \frac{v_{(\psi)}}{r\sin\theta}\frac{\partial v_{(\psi)}}{\partial \psi} + \frac{v_{(r)}v_{(\psi)}}{r} + \frac{v_{(\theta)}v_{(\psi)}\cot\theta}{r}\right]$$
$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, p_{(\psi r)}) + \frac{\partial}{\partial\theta}(r\sin\theta\, p_{(\psi\theta)}) + \frac{\partial}{\partial\psi}(r p_{(\psi\psi)})\right] + \frac{p_{(r\psi)}}{r} + \frac{p_{(\theta\psi)}\cot\theta}{r}, \tag{9}$$

where ρ is the density of the fluid. The equation of continuity is

$$\frac{\partial}{\partial r}(r^2\sin\theta\, v_{(r)}) + \frac{\partial}{\partial\theta}(r\sin\theta\, v_{(\theta)}) + \frac{\partial}{\partial\psi}(r v_{(\psi)}) = 0. \tag{10}$$

It is next necessary to characterize the elastico-viscous liquid by means of suitable equations of state. We note that when $\epsilon = 0$, the liquid is not subjected

† Brackets placed round suffices will be used to denote the physical components of vectors and tensors.

to any deformation and that the deformation is small provided ϵ is small. It is therefore possible to write the equations in the form of integral expansions (Coleman & Noll 1961; Pipkin 1964). For example, the third-order approximation is

$$p'_{ik} = \int_{-\infty}^{t} M_1(t-t')\,C_{ik}(t')\,dt' + \int_{-\infty}^{t}\int_{-\infty}^{t} M_2(t-t',t-t'')\,C_i^j(t')\,C_{jk}(t'')\,dt'\,dt''$$
$$+\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} M_3(t-t',t-t'',t-t''')\,C_i^j(t')\,C_{jl}(t'')\,C_k^l(t''')\,dt'\,dt''\,dt'''$$
$$+\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} M_4(t-t',t-t'',t-t''')\,C_l^j(t')\,C_j^l(t'')\,C_{ik}(t''')\,dt'\,dt''\,dt''', \quad (11)$$

where
$$C_{ik}(t') \equiv \frac{\partial x'^m}{\partial x^i}\frac{\partial x'^s}{\partial x^k}\,g_{ms}(\mathbf{x}') - g_{ik}(\mathbf{x}), \quad (12)$$

x'^i being the position at time t' of the element that is instantaneously at the point x^i at time t. Equation (11) could be written in a number of alternative but equivalent forms involving different measures of the deformation, but (11) has advantages from a manipulative standpoint.

Since $C_{ik} = 0$ when $\epsilon = 0$, we see that C_{ik} must be order ϵ. Our restriction to first-order terms in ϵ implies that equation (11) can now be replaced by the first-order approximation,

$$p'_{ik} = \int_{-\infty}^{t} M_1(t-t')\,C_{ik}(t')\,dt'. \quad (13)$$

In terms of the kernel function M_1 occurring in (13), the complex dynamic viscosity η^* is given by

$$\eta^* = \frac{i}{\Omega}\int_0^{\infty} M_1(\xi)\,[1-e^{-i\Omega\xi}]\,d\xi. \quad (14)$$

We shall find that η^* plays a prominent part in the analysis which follows.

One of our main objectives in the present work is to determine the couples C_x, C_y and C_z on the inner sphere. These are given in terms of the stress components by the relations,

$$C_x = -r_1^3\int_0^{2\pi}\int_0^{\pi}[p_{(r\theta)}\sin\psi + p_{(r\psi)}\cos\psi\cos\theta]\sin\theta\,d\theta\,d\psi, \quad (15)$$

$$C_y = r_1^3\int_0^{2\pi}\int_0^{\pi}[p_{(r\theta)}\cos\psi - p_{(r\psi)}\sin\psi\cos\theta]\sin\theta\,d\theta\,d\psi, \quad (16)$$

$$C_z = r_1^3\int_0^{2\pi}\int_0^{\pi} p_{(r\psi)}\sin^2\theta\,d\theta\,d\psi, \quad (17)$$

where the stresses are evaluated at $r = r_1$.

3. Solution of the equations

When the axes of rotation are coincident, i.e. when $\epsilon = 0$, all the basic equations are satisfied by

$$v_{(r)} = 0, \quad v_{(\theta)} = 0, \quad v_{(\psi)} = \Omega r\sin\theta. \quad (18)$$

Working to first order in ϵ, the boundary conditions would suggest a velocity distribution of the form,

$$\left.\begin{aligned} v_{(r)} &= \epsilon\Omega U(r,\theta)\,e^{i\psi}, \\ v_{(\theta)} &= \epsilon\Omega V(r,\theta)\,e^{i\psi}, \\ v_{(\psi)} &= \Omega r\sin\theta+\epsilon\Omega W(r,\theta)\,e^{i\psi}, \end{aligned}\right\} \tag{19}$$

where U, V and W are complex and the real part is implied.

Inspection of the deformation tensor C_{ik} indicates that it is necessary to determine the displacement functions x'^i, which we shall write as r', θ', ψ'. These are given by (Oldroyd 1950)

$$\left.\begin{aligned} \frac{\partial r'}{\partial t}+v_{(r)}\frac{\partial r'}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial r'}{\partial\theta}+\frac{v_{(\psi)}}{r\sin\theta}\frac{\partial r'}{\partial\psi} &= 0, \\ \frac{\partial \theta'}{\partial t}+v_{(r)}\frac{\partial \theta'}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial \theta'}{\partial\theta}+\frac{v_{(\psi)}}{r\sin\theta}\frac{\partial \theta'}{\partial\psi} &= 0, \\ \frac{\partial \psi'}{\partial t}+v_{(r)}\frac{\partial \psi'}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial \psi'}{\partial\theta}+\frac{v_{(\psi)}}{r\sin\theta}\frac{\partial \psi'}{\partial\psi} &= 0. \end{aligned}\right\} \tag{20}$$

Substituting (19) into (20) and solving the differential equations subject to the boundary conditions

$$r' = r, \quad \theta' = \theta, \quad \psi' = \psi \quad \text{when} \quad t' = t, \tag{21}$$

we obtain

$$\left.\begin{aligned} r' &= r-\frac{\epsilon U}{i}[1-e^{-i\Omega(t-t')}]\,e^{i\psi}, \\ \theta' &= \theta-\frac{\epsilon V}{ir}[1-e^{-i\Omega(t-t')}]\,e^{i\psi}, \\ \psi' &= \psi-\Omega(t-t')-\frac{\epsilon W}{ir\sin\theta}[1-e^{-i\Omega(t-t')}]\,e^{i\psi}. \end{aligned}\right\} \tag{22}$$

Equation (22) indicates that the particle which is at (r,θ,ψ) at the current time t will have been at that position at previous times t'', where $(t-t'') = 2n\pi/\Omega$, $n = 1, 2, \ldots$. This means that the streamlines are closed curves, which are described by individual fluid elements at the speed of rotation of the spheres.

It is next necessary to determine the metric tensor $g_{ik}(r',\theta',\psi')$. This is obtained by writing down $g_{ik}(r,\theta,\psi)$, replacing r, θ, ψ, by r', θ', ψ', respectively, and using (22). In this way, we obtain

$$\left.\begin{aligned} g_{r\theta}(r',\theta',\psi') &= g_{r\psi}(r',\theta',\psi') = g_{\theta\psi}(r',\theta',\psi') = 0, \\ g_{rr}(r',\theta',\psi') &= 1, \\ g_{\theta\theta}(r',\theta',\psi') &= r^2\left\{1-\frac{2\epsilon U}{ir}[1-e^{-i\Omega(t-t')}]\,e^{i\psi}\right\}, \\ g_{\psi\psi}(r',\theta',\psi') &= r^2\sin^2\theta\left\{1-\frac{2\epsilon[U+V\cot\theta]}{ir}[1-e^{-i\Omega(t-t')}]\,e^{i\psi}\right\}. \end{aligned}\right\} \tag{23}$$

From (12), (22) and (23), we obtain

$$\left.\begin{aligned}
C_{rr}(r',\theta',\psi') &= 2\epsilon i\frac{\partial U}{\partial r}[1-e^{-i\Omega(t-t')}]\,e^{i\psi},\\
C_{\theta\theta}(r',\theta',\psi') &= 2\epsilon ir\left[\frac{\partial V}{\partial\theta}+U\right][1-e^{-i\Omega(t-t')}]\,e^{i\psi},\\
C_{\psi\psi}(r',\theta',\psi') &= 2\epsilon ir^2\sin^2\theta\left[\frac{iW}{r\sin\theta}+\frac{(U+V\cot\theta)}{r}\right][1-e^{-i\Omega(t-t')}]\,e^{i\psi},\\
C_{r\theta}(r',\theta',\psi') &= \epsilon i\left[\frac{\partial U}{\partial\theta}+r^2\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right][1-e^{-i\Omega(t-t')}]\,e^{i\psi},\\
C_{r\psi}(r',\theta',\psi') &= \epsilon ir\sin\theta\left[\frac{iU}{r\sin\theta}+r\frac{\partial}{\partial r}\left(\frac{W}{r}\right)\right][1-e^{-i\Omega(t-t')}]\,e^{i\psi},\\
C_{\theta\psi}(r',\theta',\psi') &= \epsilon ir^2\sin\theta\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{W}{\sin\theta}\right)+\frac{iV}{r\sin\theta}\right][1-e^{-i\Omega(t-t')}]\,e^{i\psi}.
\end{aligned}\right\}\tag{24}$$

We note from (24), that although the flow is steady in the sense that $\partial/\partial t\equiv 0$, each material element is subjected to a sinusoidal deformation.

From (13) and (24), we have

$$\left.\begin{aligned}
p'_{(rr)} &= 2\Omega\epsilon\eta^*\frac{\partial U}{\partial r}e^{i\psi},\\
p'_{(\theta\theta)} &= 2\Omega\epsilon\eta^*\left[\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{U}{r}\right]e^{i\psi},\\
p'_{(\psi\psi)} &= 2\Omega\epsilon\eta^*\left[\frac{iW}{r\sin\theta}+\frac{U+V\cot\theta}{r}\right]e^{i\psi},\\
p'_{(r\theta)} &= \Omega\epsilon\eta^*\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right]e^{i\psi},\\
p'_{(r\psi)} &= \Omega\epsilon\eta^*\left[\frac{iU}{r\sin\theta}+r\frac{\partial}{\partial r}\left(\frac{W}{r}\right)\right]e^{i\psi},\\
p'_{(\theta\psi)} &= \Omega\epsilon\eta^*\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{W}{\sin\theta}\right)+\frac{iV}{r\sin\theta}\right]e^{i\psi},
\end{aligned}\right\}\tag{25}$$

where η^* is given by (14).

Substituting (19) and (25) into (7)–(9), we obtain, on equating first-order terms in ϵ,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)-\frac{U}{r^2\sin^2\theta}+\frac{2}{r^3}\frac{\partial}{\partial r}(r^2U)$$
$$-\frac{2U}{r^2}-\frac{\partial\overline{p}}{\partial r}=-\alpha^2[U+2iW\sin\theta],\tag{26}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)-\frac{2V}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial U}{\partial\theta}$$
$$-\frac{2iW\cos\theta}{r^2\sin^2\theta}-\frac{1}{r}\frac{\partial\overline{p}}{\partial\theta}=-\alpha^2[V+2iW\cos\theta],\tag{27}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial W}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial W}{\partial\theta}\right)-\frac{2W}{r^2\sin^2\theta}+\frac{2iU}{r^2\sin\theta}$$
$$+\frac{2iV\cos\theta}{r^2\sin^2\theta}-\frac{i\overline{p}}{r\sin\theta}=-\alpha^2[W-2i(U\sin\theta+V\cos\theta)], \qquad (28)$$

where
$$p=p^{(0)}+\frac{\rho\Omega^2r^2}{2}\sin^2\theta+\eta^*\epsilon\Omega\, e^{i\psi}\overline{p},$$

$p^{(0)}$ being a constant, and
$$\alpha^2=-i\Omega\rho/\eta^*. \qquad (29)$$

It is interesting to note that the parameter α^2 given by (29) also occurs in the theory for the *oscillatory* flows which are usually involved in the experimental determination of η^* (cf. Walters 1968).

Equations (26)–(28) are essentially the Navier–Stokes equations with the (frequency dependent) complex viscosity η^* replacing the (constant) Newtonian viscosity coefficient. In the present paper, we shall obtain a solution by expanding the velocity components and the pressure in powers of α^2, i.e. we write

$$\left.\begin{aligned}U&=u_0+\alpha^2u_1+\alpha^4u_2+\dots,\\V&=v_0+\alpha^2v_1+\alpha^4v_2+\dots,\\W&=w_0+\alpha^2w_1+\alpha^4w_2+\dots,\\\overline{p}&=\overline{p}_0+\alpha^2\overline{p}_1+\alpha^4\overline{p}_2+\dots.\end{aligned}\right\} \qquad (30)$$

(i) *Zero-order solution*

We obtain first the solution for $\alpha^2=0$, which corresponds to the case when the effect of fluid inertia may be ignored. This zero-order solution is of some practical importance, since the neglect of fluid inertia would be justified in many applications of the Balance Rheometer.

The boundary conditions (6) suggest a zero-order velocity distribution of the form,
$$u_0=0,\quad v_0=if(r),\quad w_0=-f(r)\cos\theta, \qquad (31)$$

which automatically satisfies the equation of continuity (10). Substituting (31) into (26)–(28), with $\alpha^2=0$, we obtain

$$\overline{p}_0=0 \qquad (32)$$

and
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)-\frac{2f}{r^2}=0. \qquad (33)$$

The solution of (33) subject to
$$f=0\quad\text{on}\quad r=r_1,$$
$$f=r_2\quad\text{on}\quad r=r_2,$$

is
$$f=\lambda[r-(r_1^3/r^2)], \qquad (34)$$

where
$$\lambda=r_2^3/(r_2^3-r_1^3). \qquad (35)$$

We substitute (31), (34) and (35) into the stress components (25), take the real parts of these components and use (15)–(17), to obtain the zero-order couples $C_x^{(0)}$, $C_y^{(0)}$, $C_z^{(0)}$ in the form,

$$\left.\begin{aligned} C_x^{(0)} &= 8\pi\eta'\Omega\epsilon\lambda r_1^3, \\ C_y^{(0)} &= 8\pi G'\epsilon\lambda r_1^3, \\ C_z^{(0)} &= 0. \end{aligned}\right\} \tag{36}$$

The analysis of this section formalises the earlier work of Jones & Walters (1969) who noted that the solution for a Newtonian viscous liquid, which can be obtained very simply from the superposition principle, also applies to an elastico-viscous liquid.

We note from (36) that experimental $(C_x^{(0)}, \Omega)$ and $(C_y^{(0)}, \Omega)$ results can be immediately converted into meaningful (η', Ω) and (G', Ω) data. This indicates that the Balance Rheometer can be used to determine the dynamic viscosity and the dynamic rigidity in cases when fluid inertia is negligible. In order to determine to what extent this interpretation is modified when fluid inertia cannot be ignored, it is necessary to proceed to higher-order approximations.

(ii) *First-order solution*

Substituting (31) and (34) into (26)–(30) and equating terms involving α^2, we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_1}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_1}{\partial\theta}\right)-\frac{u_1}{r^2\sin^2\theta}+\frac{2}{r^3}\frac{\partial}{\partial r}(r^2u_1)$$
$$-\frac{2u_1}{r^2}-\frac{\partial\bar{p}_1}{\partial r}=2i\lambda\sin\theta\cos\theta\left[r-\frac{r_1^3}{r^2}\right], \tag{37}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_1}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_1}{\partial\theta}\right)-\frac{2v_1}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial u_1}{\partial\theta}$$
$$-\frac{2iw_1\cos\theta}{r^2\sin^2\theta}-\frac{1}{r}\frac{\partial\bar{p}_1}{\partial\theta}=i\lambda[2\cos^2\theta-1]\left[r-\frac{r_1^3}{r^2}\right], \tag{38}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w_1}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w_1}{\partial\theta}\right)-\frac{2w_1}{r^2\sin^2\theta}+\frac{2iu_1}{r^2\sin\theta}$$
$$+\frac{2iv_1\cos\theta}{r^2\sin^2\theta}-\frac{i\bar{p}_1}{r\sin\theta}=-\lambda\cos\theta\left[r-\frac{r_1^3}{r^2}\right]. \tag{39}$$

These equations have to be solved subject to

$$u_1 = v_1 = w_1 = 0 \quad \text{on} \quad r = r_1 \quad \text{and} \quad r = r_2. \tag{40}$$

The work of Walters & Waters (1963)† and the form of the forcing functions in (37)–(39) strongly suggest a velocity distribution of the form

$$\left.\begin{aligned} u_1 &= -\frac{6iF(r)}{r^2}\sin\theta\cos\theta, \\ v_1 &= \frac{i}{r}\frac{dF}{dr}[\sin^2\theta-\cos^2\theta], \\ w_1 &= \frac{1}{r}\frac{dF}{dr}\cos\theta, \end{aligned}\right\} \tag{41}$$

† Walters & Waters (1963) considered the related problem when the fluid is contained between spheres which rotate about the *same* axis with *different* angular velocities.

which automatically satisfies the equation of continuity (10). Substituting (41) into (37)–(39), and writing $\bar{p}_1 = i\bar{\bar{p}}_1(r)\sin\theta\cos\theta$, we obtain

$$-\frac{6}{r^2}\frac{d^2F}{dr^2}+\frac{36F}{r^4}-\frac{d\bar{\bar{p}}_1}{dr} = 2\lambda\left[r-\frac{r_1^3}{r^2}\right] \tag{42}$$

from the first equation, and

$$\bar{\bar{p}}_1 = -\lambda\left[r-\frac{r_1^3}{r^2}\right]-\left[\frac{d^3F}{dr^3}-\frac{6}{r}\frac{dF}{dr}+\frac{12F}{r^3}\right] \tag{43}$$

from both the second and third equations. Eliminating $\bar{\bar{p}}_1$, we have finally

$$r^4\frac{d^4F}{dr^4}-12r^2\frac{d^2F}{dr^2}+24r\frac{dF}{dr} = -3\lambda r_1^3 r^2, \tag{44}$$

which has to be solved subject to

$$F = \frac{dF}{dr} = 0 \quad \text{on} \quad r = r_1 \quad \text{and} \quad r = r_2. \tag{45}$$

The solution of (44) subject to (45) is (cf. Walters & Waters 1963)

$$F = A+(B/r^2)+Cr^3+Dr^5+Er^2, \tag{46}$$

where

$$\left.\begin{aligned} A &= \frac{\lambda r_1^3 r_2^2}{8\Delta}[8-15\beta+7\beta^3+7\beta^5-15\beta^7+8\beta^8],\\ B &= -\frac{\lambda r_1^3 r_2^4}{8\Delta}[4-9\beta+10\beta^3-9\beta^5+4\beta^6],\\ C &= \frac{\lambda r_1^2}{8\Delta}[6-20\beta^2+14\beta^4+14\beta^5-20\beta^7+6\beta^9],\\ D &= -\frac{\lambda}{4\Delta}[1-6\beta^2+5\beta^3+5\beta^4-6\beta^5+\beta^7],\\ E &= -\tfrac{1}{8}\lambda r_1^3, \end{aligned}\right\} \tag{47}$$

and

$$\left.\begin{aligned} \beta &= r_2/r_1,\\ \Delta &= [4-25\beta^3+42\beta^5-25\beta^7+4\beta^{10}]. \end{aligned}\right\} \tag{48}$$

From (15)–(17), (25), (30), (41) and (46)–(48), it is not difficult to show that the couples to order α^2 are all zero, i.e.

$$\left.\begin{aligned} C_x^{(2)} &= 0,\\ C_y^{(2)} &= 0,\\ C_z^{(2)} &= 0. \end{aligned}\right\} \tag{49}$$

This means that terms of order α^4 need to be considered to assess the effect of fluid inertia on the couples C_x, C_y and C_z.

(iii) *Second-order solution*

Substituting (41), (46)–(48) into the equations of motion (26)–(28) and equating powers of α^4, we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_2}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_2}{\partial\theta}\right)-\frac{u_2}{r^2\sin^2\theta}$$
$$+\frac{2}{r^3}\frac{\partial}{\partial r}(r^2u_2)-\frac{2u_2}{r^2}-\frac{\partial\bar{p}_2}{\partial r}=\left[\frac{6F}{r^2}-\frac{2}{r}\frac{dF}{dr}\right]i\sin\theta\cos\theta, \tag{50}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_2}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_2}{\partial\theta}\right)-\frac{2v_2}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial u_2}{\partial\theta}$$
$$-\frac{2iw_2\cos\theta}{r^2\sin^2\theta}-\frac{1}{r}\frac{\partial\bar{p}_2}{\partial\theta}=-\frac{i}{r}\frac{dF}{dr}, \tag{51}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w_2}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w_2}{\partial\theta}\right)-\frac{2w_2}{r^2\sin^2\theta}+\frac{2iu_2}{r^2\sin\theta}$$
$$+\frac{2iv_2\cos\theta}{r^2\sin^2\theta}-\frac{i\bar{p}_2}{r\sin\theta}=\frac{1}{r}\frac{dF}{dr}\cos\theta+\left[\frac{12F}{r^2}-\frac{4}{r}\frac{dF}{dr}\right]\sin^2\theta\cos\theta, \tag{52}$$

where F is given by (46).

Close inspection of the forcing functions in (50)–(52) suggests that we write

$$\left.\begin{aligned} u_2 &= iH(r)\sin\theta\cos\theta,\\ v_2 &= iJ(r)+iK(r)\sin^2\theta,\\ w_2 &= -J(r)\cos\theta+L(r)\sin^2\theta\cos\theta,\\ \bar{p}_2 &= i\bar{\bar{p}}_2(r)\sin\theta\cos\theta. \end{aligned}\right\} \tag{53}$$

The equation of continuity requires

$$\frac{1}{r}\frac{d}{dr}(r^2H)+3K+L=0. \tag{54}$$

Substituting (53) into (50)–(52), we obtain

$$\frac{d^2H}{dr^2}+\frac{4}{r}\frac{dH}{dr}-\frac{4H}{r^2}-\frac{d\bar{\bar{p}}_2}{dr}=\frac{6F}{r^2}-\frac{2}{r}\frac{dF}{dr}, \tag{55}$$

$$\frac{d^2K}{dr^2}+\frac{2}{r}\frac{dK}{dr}-\frac{6K}{r^2}-\frac{4H}{r^2}+\frac{2L}{r^2}=-\frac{2\bar{\bar{p}}_2}{r}, \tag{56}$$

$$\frac{d^2L}{dr^2}+\frac{2}{r}\frac{dL}{dr}-\frac{12L}{r^2}=\frac{12F}{r^2}-\frac{4}{r}\frac{dF}{dr}, \tag{57}$$

$$\frac{d^2J}{dr^2}+\frac{2}{r}\frac{dJ}{dr}-\frac{2J}{r^2}=\frac{\bar{\bar{p}}_2}{r}-\frac{1}{r}\frac{dF}{dr}+\frac{2}{r^2}[L-K-H]. \tag{58}$$

Substituting (56) into (55) and using (54), we obtain

$$r^4\frac{d^4H}{dr^4}+8r^3\frac{d^3H}{dr^3}-24r\frac{dH}{dr}+24H$$
$$= 12r\frac{dF}{dr}-36F-\left[r^3\frac{d^3L}{dr^3}+3r^2\frac{d^2L}{dr^2}-12r\frac{dL}{dr}+12L\right]. \tag{59}$$

From (57) and (59), we have finally

$$r^4\frac{d^4H}{dr^4}+8r^3\frac{d^3H}{dr^3}-24r\frac{dH}{dr}+24H = 4r^2\frac{d^2F}{dr^2}-24F, \tag{60}$$

which has to be solved subject to

$$H = dH/dr = 0 \quad \text{on} \quad r = r_1 \quad \text{and} \quad r = r_2. \tag{61}$$

The solution of (60) subject to (61) is

$$H = -A+\tfrac{2}{3}Er^2+\frac{Dr^5}{9}+Pr+\frac{Q}{r^2}+Rr^3+\frac{S}{r^4}, \tag{62}$$

$$\left.\begin{aligned}
P &= \frac{1}{\Delta}\left[\frac{A}{r_1}P_1(\beta)-\tfrac{4}{3}Er_1P_2(\beta)+\tfrac{1}{9}Dr_1^4P_3(\beta)\right],\\
Q &= \frac{1}{\Delta}[Ar_2^2Q_1(\beta)+\tfrac{2}{3}Er_1r_2^3Q_2(\beta)-\tfrac{1}{9}Dr_1^4r_2^3Q_3(\beta)],\\
R &= \frac{1}{\Delta}\left[-\frac{A}{r_1^3}R_1(\beta)-\frac{4}{3}\frac{E}{r_1}R_2(\beta)-\tfrac{1}{9}Dr_1^2R_3(\beta)\right],\\
S &= \frac{1}{\Delta}[-Ar_2^4S_1(\beta)-\tfrac{2}{3}Er_1r_2^5S_2(\beta)+\tfrac{1}{9}Dr_1^4r_2^5S_3(\beta)].
\end{aligned}\right\} \tag{63}$$

A, E, D and Δ are given by (47) and (48), and

$$\left.\begin{aligned}
P_1 &= 6-20\beta^2+14\beta^4+14\beta^5-20\beta^7+6\beta^9,\\
Q_1 &= 8-15\beta+7\beta^3+7\beta^5-15\beta^7+8\beta^8,\\
R_1 &= 2-12\beta^2+10\beta^3+10\beta^4-12\beta^5+2\beta^7,\\
S_1 &= 4-9\beta+10\beta^3-9\beta^5+4\beta^6,\\
P_2 &= 1-15\beta^4+14\beta^5+14\beta^6-15\beta^7+\beta^{11},\\
Q_2 &= 5-12\beta+7\beta^2+7\beta^6-12\beta^7+5\beta^8,\\
R_2 &= 1-10\beta^3+9\beta^4+9\beta^5-10\beta^6+\beta^9,\\
S_2 &= 3-8\beta+5\beta^2+5\beta^4-8\beta^5+3\beta^6,\\
P_3 &= 4-49\beta^5+90\beta^7-49\beta^9+4\beta^{14},\\
Q_3 &= 10-28\beta^2+18\beta^4+18\beta^7-28\beta^9+10\beta^{11},\\
R_3 &= 8-35\beta^3+27\beta^5+27\beta^7-35\beta^9+8\beta^{12},\\
S_3 &= 6-20\beta^2+14\beta^4+14\beta^5-20\beta^7+6\beta^9.
\end{aligned}\right\} \tag{64}$$

The solution of (57), subject to

$$L = 0 \quad \text{on} \quad r = r_1 \quad \text{and} \quad r = r_2, \tag{65}$$

is

$$L = Tr^3 + \frac{U}{r^4} - \left[A + \frac{2B}{r^2} + \tfrac{4}{9}Dr^5 + \tfrac{2}{3}Er^2\right], \tag{66}$$

where

$$\left.\begin{aligned} T &= \frac{1}{\beta^7 - 1}\left[\frac{A}{r_1^3}(\beta^4 - 1) + \frac{2B}{r_1^5}(\beta^2 - 1) + \tfrac{4}{9}Dr_1^2(\beta^9 - 1) + \frac{2E}{3r_1}(\beta^6 - 1)\right], \\ U &= \frac{1}{\beta^7 - 1}[Ar_2^4(\beta^3 - 1) + 2Br_2^2(\beta^5 - 1) + \tfrac{4}{9}Dr_1^2 r_2^7(1 - \beta^2) + \tfrac{2}{3}Er_2^6(\beta - 1)]. \end{aligned}\right\} \tag{67}$$

From (54), we obtain

$$K = A + \frac{2B}{3r^2} - \tfrac{1}{9}Dr^5 - \tfrac{2}{3}Er^2 - Pr - \tfrac{1}{3}(5R + T)r^3 + \tfrac{1}{3}(2S - U)\frac{1}{r^4}, \tag{68}$$

and from (56)

$$\bar{\bar{p}}_2 = \frac{2A}{r} + \frac{10B}{3r^3} + 2Dr^4 + 2Er + \frac{2Q}{r^3} + 7Rr^2. \tag{69}$$

Substituting (46), (62), (66), (68) and (69) into (58), we have

$$r^2\frac{d^2J}{dr^2} + 2r\frac{dJ}{dr} - 2J = [\tfrac{25}{3}R + \tfrac{8}{3}T - 3C]\,r^3 - \tfrac{35}{9}Dr^5 - \tfrac{4}{3}Er^2 + [\tfrac{8}{3}U - \tfrac{10}{3}S]\,(1/r^4). \tag{70}$$

The solution of (70) subject to

$$J = 0 \quad \text{on} \quad r = r_1 \quad \text{and} \quad r = r_2 \tag{71}$$

is

$$J = Yr + \frac{Z}{r^2} - \frac{5D}{36}r^5 - \frac{E}{3}r^2 + Wr^3 + \frac{X}{r^4}, \tag{72}$$

where

$$\left.\begin{aligned} W &= \tfrac{1}{10}[\tfrac{25}{3}R + \tfrac{8}{3}T - 3C], \\ X &= \tfrac{1}{30}[8U - 5S], \\ Y &= \frac{1}{\beta^3 - 1}\left[\frac{5Dr_1^4}{36}(\beta^7 - 1) + \frac{E}{3}r_1(\beta^4 - 1) + Wr_1^2(1 - \beta^5) + \frac{X}{r_1^3 r_2^2}(\beta^2 - 1)\right], \\ Z &= \frac{1}{\beta^3 - 1}\left[\frac{5D}{36}r_2^3 r_1^4(1 - \beta^4) + \frac{E}{3}r_2^3 r_1(1 - \beta) + Wr_1^2 r_2^3(\beta^2 - 1) + \frac{X}{r_2^2}(1 - \beta^5)\right]. \end{aligned}\right\} \tag{73}$$

From (15) to (17), (30) and (53), it can be shown that the couples to order α^4 ($C_x^{(4)}$, $C_y^{(4)}$, $C_z^{(4)}$) are given by

$$\left.\begin{aligned} C_x^{(4)} &= \tfrac{4}{15}\pi\epsilon r_1^3(\Omega\eta'\alpha_R^4 + G'\alpha_I^4)\,G(r_1), \\ C_y^{(4)} &= \tfrac{4}{15}\pi\epsilon r_1^3(G'\alpha_R^4 - \eta'\Omega\alpha_I^4)\,G(r_1), \\ C_z^{(4)} &= 0, \end{aligned}\right\} \tag{74}$$

where

$$\alpha^4 = \alpha_R^4 + i\alpha_I^4 \tag{75}$$

and

$$G(r) = 10r\frac{d}{dr}\left(\frac{J}{r}\right) + 5r\frac{d}{dr}\left(\frac{K}{r}\right) - r\frac{d}{dr}\left(\frac{L}{r}\right). \tag{76}$$

4. Discussion

We see from (35), (36), (49), (74) and (76) that the operating formula for a given balance rheometer (to order α^4) can be obtained by computing λ and $G(r_1)$ corresponding to the dimensions of the instrument.

We are particularly interested in the effect of fluid inertia on the interpretation of experimental results. In order to assess this in the most convenient way we determine $G(r_1)$ for very small gaps by writing $r = r_1 + d\bar{x}$ and working to the leading power in d. In this way, we obtain

$$\left.\begin{aligned} F &= -\tfrac{1}{8}\lambda r_1 d^4[\bar{x}^4 - 2\bar{x}^3 + \bar{x}^2], \\ H &= -(\lambda d^6/120 r_1)[2\bar{x}^6 - 6\bar{x}^5 + 5\bar{x}^4 - \bar{x}^2], \\ L &= (\lambda d^5/60)[6\bar{x}^5 - 15\bar{x}^4 + 10\bar{x}^3 - \bar{x}], \\ K &= O(d^6), \\ \bar{\bar{p}}_2 &= O(d^4), \\ J &= (\lambda d^5/240)[6\bar{x}^5 - 15\bar{x}^4 + 10\bar{x}^3 - \bar{x}]. \end{aligned}\right\} \tag{77}$$

From (76) and (77), we obtain

$$G(r_1) = -(\lambda d^4/40) \tag{78}$$

and

$$\begin{aligned} C_x^{(4)} &= -(\pi\epsilon r_1^3\lambda/150)[\Omega\eta'\alpha_R^4 + G'\alpha_I^4]\,d^4, \\ C_y^{(4)} &= (\pi\epsilon r_1^3\lambda/150)[\Omega\eta'\alpha_I^4 - G'\alpha_R^4]\,d^4, \end{aligned} \tag{79}$$

where terms of order d^5 have been ignored. From (36) and (79), we can write

$$C_x - iC_y = 8\pi\epsilon\lambda r_1^3\Omega\eta^*[1 - (d^4\alpha^4/1200)], \tag{80}$$

where terms of order α^6 have been ignored.

For the Balance Rheometer of Kepes $d = 0{\cdot}1$ cm, and it will be seen from (80) that the 'inertia' correction in this case is extremely small. We conclude that inertial effects are likely to be negligible in most applications of the Balance Rheometer, and that (80) may be used to take these into account in extreme cases of low viscosity fluids and high rotational speeds.

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